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ON THE ASYMPTOTIC DIRICHLET PROBLEM FOR THE MINIMAL HYPERSURFACE EQUATION IN A HADAMARD MANIFOLD

JEAN-BAPTISTE CASTERAS, ILKKA HOLOPAINEN, AND JAIME B. RIPOLL

ABSTRACT. We study the Dirichlet problem at infinity on a Cartan-Hadamard manifold M of dimension $n \geq 2$ for a large class of operators containing, in particular, the p -Laplacian and the minimal graph operator. We extend several existence results obtained for the p -Laplacian to our class of operators. As an application of our main result, we prove the solvability of the asymptotic Dirichlet problem for the minimal graph equation for any continuous boundary data on a (possibly non rotationally symmetric) manifold whose sectional curvatures are allowed to decay to 0 quadratically.

1. INTRODUCTION

In this paper we study the asymptotic Dirichlet problem for operators

$$(1.1) \quad \mathcal{Q}[u] := \operatorname{div} \mathcal{A}(|\nabla u|^2) \nabla u$$

on Cartan-Hadamard manifolds where \mathcal{A} is a non-negative function in $[0, +\infty)$ subject to some growth conditions. Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n -manifold, $n \geq 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o: T_o M \rightarrow M$ is a diffeomorphism for every point $o \in M$. Consequently, M is diffeomorphic to \mathbb{R}^n . A Cartan-Hadamard manifold M can be compactified by adding a *sphere at infinity*, denoted by $M(\infty)$, so that the resulting space $\bar{M} = M \cup M(\infty)$ equipped with the so-called *cone topology* is homeomorphic to a closed Euclidean ball; see [15]. The *asymptotic Dirichlet problem* on M for the operator \mathcal{Q} is then the following: Given a continuous function h on $M(\infty)$ does there exist a (unique) function $u \in C(\bar{M})$ such that $\mathcal{Q}[u] = 0$ on M and $u|_{M(\infty)} = h$?

We assume that $\mathcal{A}: (0, \infty) \rightarrow [0, \infty)$ is a smooth function such that

$$(1.2) \quad \mathcal{A}(t) \leq A_0 t^{(p-2)/2}$$

for all $t > 0$ and for some constants $A_0 > 0$ and $p \geq 1$, and that $\mathcal{B} := \mathcal{A}'/\mathcal{A}$ satisfies

$$(1.3) \quad -\frac{1}{2t} < \mathcal{B}(t) \leq \frac{B_0}{t}$$

for all $t > 0$ and for some constant $B_0 > -1/2$. Furthermore, we assume that $t\mathcal{A}(t^2) \rightarrow 0$ as $t \rightarrow 0+$ and therefore we set $\mathcal{A}(|X|^2)X = 0$ whenever X is a zero vector. As a consequence of (1.3), the function $t \mapsto t\mathcal{A}(t^2)$ is strictly increasing. A

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function u is a (weak) solution to the equation $\mathcal{Q}[u] = 0$ in an open set $\Omega \subset M$ if it belongs to the local Sobolev space $W_{\text{loc}}^{1,p}(\Omega)$ and

$$(1.4) \quad \int_{\Omega} \langle \mathcal{A}(|\nabla u|^2) \nabla u, \nabla \varphi \rangle dm = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Here and in what follows the integration is with respect to the Riemannian volume form dm on M . Such function u will be called a \mathcal{Q} -solution in Ω . Furthermore, we say that a function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a \mathcal{Q} -subsolution in Ω if $\mathcal{Q}[u] \geq 0$ weakly in Ω , that is

$$(1.5) \quad \int_{\Omega} \langle \mathcal{A}(|\nabla u|^2) \nabla u, \nabla \varphi \rangle dm \leq 0$$

for every non-negative $\varphi \in C_0^\infty(U)$. Similarly, a function $v \in W_{\text{loc}}^{1,p}(U)$ is called a \mathcal{Q} -supersolution in Ω if $-v$ is a \mathcal{Q} -subsolution in Ω . Note that $u + c$ is a \mathcal{Q} -solution (respectively, \mathcal{Q} -subsolution, \mathcal{Q} -supersolution) for every constant c if u is a \mathcal{Q} -solution (respectively, \mathcal{Q} -subsolution, \mathcal{Q} -supersolution). It follows from the growth condition (1.2) that test functions φ in (1.4) and (1.5) can be taken from the class $W_0^{1,p}(\Omega)$ if $|\nabla u| \in L^p(\Omega)$.

We call a relatively compact open set $\Omega \Subset M$ \mathcal{Q} -regular if for any continuous boundary data $h \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega})$ which is a \mathcal{Q} -solution in Ω and $u|_{\partial\Omega} = h$. In addition to the growth conditions on \mathcal{A} , we assume that $\mathcal{Q} = \mathcal{Q}_{\mathcal{A},M}$ is such that

- (A) there is an exhaustion of M by an increasing sequence of \mathcal{Q} -regular domains Ω_k , and that
- (B) any locally uniformly bounded sequence of continuous \mathcal{Q} -solutions contains a locally uniformly convergent subsequence.

In this paper the primary example of the equations that satisfy the conditions above is the *minimal graph equation*

$$(1.6) \quad \mathcal{M}[u] := \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0,$$

in which case

$$\mathcal{A}(t) = \frac{1}{\sqrt{1+t}} \text{ and } \mathcal{B}(t) = -\frac{1}{2(1+t)},$$

and therefore (1.2) and (1.3) hold with constants $A_0 = 1$, $p = 1$, and $B_0 = 0$, respectively. We note that u satisfies (1.6) if and only if $G := \{(x, u(x)) : x \in \Omega\}$ is a minimal hypersurface in $M \times \mathbb{R}$. For the minimal graph equation, condition (A) follows from [13, Theorem 2] where Ω_k may be chosen as a geodesic ball with radius k centered at a fixed point of M , and condition (B) follows from [29, Theorem 1.1] (see also [13, Theorem 1]).

The class of equations considered here includes also the p -Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty,$$

in which case

$$\mathcal{A}(t) = t^{(p-2)/2} \text{ and } \mathcal{B}(t) = \frac{p-2}{2t},$$

and so $A_0 = 1$ and $B_0 = (p-2)/2$. In the special case $p = 2$ one obtains the usual Laplace-Beltrami equation $\Delta u = 0$, with $\mathcal{A}(t) \equiv 1$ and $\mathcal{B}(t) \equiv 0$. It is well-known that the properties (A) and (B) above hold for the p -Laplace equation and that

(weak) solutions of the p -Laplace equation have Hölder-continuous representatives, usually called p -harmonic functions; see [17].

The asymptotic Dirichlet problem for the Laplace-Beltrami operator was solved affirmatively by Choi [10] under assumptions that sectional curvatures satisfy $\text{Sect} \leq -a^2 < 0$ and any two points in $M(\infty)$ can be separated by convex neighborhoods. Such appropriate convex sets were constructed by Anderson [5] for manifolds of pinched sectional curvature $-b^2 \leq \text{Sect} \leq -a^2 < 0$. Independently, Sullivan [30] solved the Dirichlet problem at infinity under the same pinched curvature assumption by using probabilistic arguments. In [6], Anderson and Schoen presented a simple and direct solution to the Dirichlet problem again in the case of pinched negative curvature. By modifying Anderson's argument, Borbély [7] was able to construct appropriate convex sets under a weaker curvature lower bound $\text{Sect}_x \geq -g(\rho(x))$, where $g(t) \approx e^{\lambda t}$, with $\lambda < 1/3$. Here and throughout the paper $\rho(x)$ stands for the distance between $x \in M$ and a fixed point $o \in M$. Major contributions to the Dirichlet problem were given by Ancona in a series of papers [1], [2], [3], and [4]. In particular, he was able to replace the curvature lower bound with a bounded geometry assumption that each ball up to a fixed radius is L -bi-Lipschitz equivalent to an open set in \mathbb{R}^n for some fixed $L \geq 1$; see [1]. On the other hand, in [4] Ancona constructed a 3-dimensional Cartan-Hadamard manifold with sectional curvatures bounded from above by -1 where the asymptotic Dirichlet problem is not solvable. Another example of a (3-dimensional) Cartan-Hadamard manifold, with sectional curvatures ≤ -1 , on which the asymptotic Dirichlet problem is not solvable was constructed by Borbély [8]. To the best of our knowledge, the most general curvature bounds under which the asymptotic Dirichlet problem for the Laplace-Beltrami equation is solvable are given in the following theorems by Hsu (see also Theorems 1.3 and 1.4 below).

Theorem 1.1. [22, Theorem 1.1] *Let M be a Cartan-Hadamard manifold. Suppose that there exist a positive constant a and a positive and non-increasing function λ with $\int_0^\infty t\lambda(t) dt < \infty$ such that*

$$-\lambda(\rho(x))^2 e^{2a\rho(x)} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect} \leq -a^2.$$

Then the Dirichlet problem at infinity of M for the Laplace-Beltrami equation is solvable.

Theorem 1.2. [22, Theorem 1.2] *Let M be a Cartan-Hadamard manifold. Suppose that there exist positive constants r_0 , $\alpha > 2$, and $\beta < \alpha - 2$ such that*

$$-\rho(x)^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -\frac{\alpha(\alpha-1)}{\rho(x)^2}$$

for all $x \in M$, with $\rho(x) \geq r_0$. Then the Dirichlet problem at infinity of M for the Laplace-Beltrami equation is solvable.

The asymptotic Dirichlet problem has been studied also in a more general context of p -harmonic and \mathcal{A} -harmonic functions as well as for operators \mathcal{Q} . For the p -Laplace equation the asymptotic Dirichlet problem was solved in [19] on Cartan-Hadamard manifolds of pinched negative sectional curvature by modifying the direct approach of Anderson and Schoen [6]. In [21] Holopainen and Vähäkangas studied the asymptotic Dirichlet problem for the p -Laplace equation and the p -regularity of a point x_0 at infinity on a Cartan-Hadamard manifold M under a

curvature assumption

$$-b(\rho(x))^2 \leq \text{Sect}_x \leq -a(\rho(x))^2$$

in $U \cap M$, where U is a neighborhood of $x_0 \in M(\infty)$. Here $a, b: [0, \infty) \rightarrow [0, \infty)$, $b \geq a$, are smooth functions subject to certain growth conditions; see Section 2. The following two special cases of functions a and b are of particular interest.

Theorem 1.3. [21, Corollary 3.22] *Let $\phi > 1$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$(1.7) \quad -\rho(x)^{2\phi-4-\varepsilon} \leq \text{Sect}_x \leq -\frac{\phi(\phi-1)}{\rho(x)^2}$$

for every $x \in U \cap M$. Then x_0 is a p -regular point at infinity for every $p \in (1, 1 + (n-1)\phi)$.

Theorem 1.4. [21, Corollary 3.23] *Let $k > 0$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$(1.8) \quad -\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq \text{Sect}_x \leq -k^2$$

for every $x \in U \cap M$. Then x_0 is a p -regular point at infinity for every $p \in (1, \infty)$.

Roughly speaking, the p -regularity of $x_0 \in M(\infty)$ means that, at the point x_0 , the Dirichlet problem for the p -Laplace equation is solvable with continuous boundary data; see [21] and [32] for the details. In particular, the Dirichlet problem at infinity for the p -Laplace equation is solvable if every point $x_0 \in M(\infty)$ is p -regular. The case of the usual Laplacian ($p = 2$) is covered by Theorem 1.3 for every $\phi > 1$ since then $1 + (n-1)\phi > 2$. Thus the assumptions in Theorem 1.3 are slightly weaker than those in Theorem 1.2. Note that using the Ricci curvature instead of the sectional makes no essential difference since all sectional curvatures are nonpositive. On the other hand, Theorem 1.4 and Theorem 1.1 are closely related in the case $p = 2$ but, nevertheless, slightly different and neither one implies the other directly.

In [32] Vähäkangas generalized the method and results due to Cheng [9] and showed that $x_0 \in M(\infty)$ is p -regular if it has a neighborhood V in the cone topology such that the radial sectional curvatures in $V \cap M$ satisfy a pointwise pinching condition

$$|\text{Sect}_x(P)| \leq C|\text{Sect}_x(P')|$$

for some constant C and have an upper bound

$$\text{Sect}_x(P) \leq -\frac{\phi(\phi-1)}{\rho^2(x)}$$

for some constant $\phi > 1$ with $1 < p < 1 + \phi(n-1)$. Above P and P' are any 2-dimensional subspaces of $T_x M$ containing the (radial) vector $\nabla \rho(x)$. It is worth observing that no curvature lower bounds are needed here. In fact, Vähäkangas considered even a more general case of \mathcal{A} -harmonic functions (of type $p \in (1, \infty)$), i.e. continuous weak solutions to the equation

$$\text{div } \mathcal{A}(\nabla u) = 0,$$

where \mathcal{A} is subject to certain conditions; for instance $\langle \mathcal{A}(V), V \rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Note that this class of equations is different from ours in the current paper, although both include the p -Laplace

equation. Recently, Vähäkangas generalized Theorems 1.3 and 1.4 to cover the case of \mathcal{A} -harmonic functions as well; see [33, Corollary 3.7, Corollary 3.8, Remark 3.9].

In [11] Collin and Rosenberg constructed harmonic diffeomorphisms from the complex plane \mathbb{C} onto the hyperbolic plane \mathbb{H}^2 disproving a conjecture of Schoen and Yau [28]. A bit later Gálvez and Rosenberg [16] extended the result to any Hadamard surface M whose curvature is bounded from above by a negative constant by proving the existence of harmonic diffeomorphisms from \mathbb{C} onto M . The proofs in both papers are based on the construction of an entire minimal surface $\Sigma = (x, u(x)) \subset \mathbb{H}^2 \times \mathbb{R}$ ($\Sigma \subset M \times \mathbb{R}$, resp.) of conformal type \mathbb{C} , and thus on the construction of an entire solution u to the minimal graph equation that is unbounded both from above and from below. Harmonic diffeomorphisms $\mathbb{C} \rightarrow \mathbb{H}^2$ ($\mathbb{C} \rightarrow M$, resp.) are then obtained by composing conformal mappings (diffeomorphisms) $\mathbb{C} \rightarrow \Sigma$ with harmonic vertical projections $\Sigma \rightarrow \mathbb{H}^2$ ($\Sigma \rightarrow M$, resp.). A crucial method in the construction of an entire unbounded solution u to the minimal graph equation is to solve the Dirichlet problem on unbounded ideal polygons with boundary values $\pm\infty$ on the sides of the ideal polygons. The unexpected result of Collin and Rosenberg has raised interest in (entire) minimal hypersurfaces in the product space $M \times \mathbb{R}$, where M is a Cartan-Hadamard manifold. Motivated by the recent research in this field (see for example, [12], [14], [23], [24], [25], [26], [27], [29]) we investigate in the present paper a possible extension of the results for the p -Laplacian obtained in [21] to the minimal graph PDE.

Of particular interest is the following special case of our main theorem (Theorem 1.6).

Theorem 1.5. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Fix $o \in M$ and set $\rho(\cdot) = d(o, \cdot)$, where d is the Riemannian distance in M . Assume that*

$$(1.9) \quad -\rho(x)^{2(\phi-2)-\varepsilon} \leq \text{Sect}_x(P) \leq -\frac{\phi(\phi-1)}{\rho(x)^2},$$

for some constants $\phi > 1$ and $\varepsilon > 0$, where $\text{Sect}_x(P)$ is the sectional curvature of a plane $P \subset T_x M$ and x is any point in the complement of a ball $B(o, R_0)$. Then the asymptotic Dirichlet problem for the minimal graph equation (1.6) is uniquely solvable for any boundary data $f \in C(M(\infty))$.

So far, the solvability of the asymptotic Dirichlet problem for the minimal graph equation has been established only under hypothesis which included the condition $\text{Sect}_x(P) \leq c < 0$ (see [16], [25]). In [25] Ripoll and Telichevesky introduced the following *strict convexity condition* (SC condition) that applies to equations (1.1). A Cartan-Hadamard manifold M satisfies the strict convexity condition if, for every $x \in M(\infty)$ and relatively open subset $W \subset M(\infty)$ containing x , there exists a C^2 open subset $\Omega \subset M$ such that $x \in \text{Int}(\Omega(\infty)) \subset W$ and $M \setminus \Omega$ is convex. They proved that the asymptotic Dirichlet problem for (1.1) on M is solvable if $\text{Sect} \leq -k^2 < 0$ and M satisfies the SC condition; see [25, Theorem 7]. Furthermore, they showed by modifying Anderson's and Borbély's arguments that M satisfies the SC condition provided there exist constants $k > 0$, $\varepsilon > 0$, and R^* such that

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq \text{Sect}_x \leq -k^2$$

for all $x \in M \setminus B(o, R^*)$ thus generalizing Theorem 1.4; see [25, Theorem 14].

The main theorem of the paper is the following solvability result for the asymptotic Dirichlet problem for operators \mathcal{Q} that satisfy (1.2), (1.3), and conditions (A) and (B) under curvature assumption

$$-b(\rho(x))^2 \leq \text{Sect}_x \leq -a(\rho(x))^2$$

on M , where $a, b: [0, \infty) \rightarrow [0, \infty)$, $b \geq a$, are smooth functions satisfying assumptions (A1)-(A7) (see Section 2). The constant ϕ_1 below is related to the assumption (A1). More precisely,

$$\phi_1 = \frac{1 + \sqrt{1 + 4C_1^2}}{2} > 1,$$

where $C_1 > 0$ is a constant such that, for all $t \geq T_1 > 0$,

$$a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \geq C_1 t^{-1} & \text{if } b \text{ is increasing.} \end{cases}$$

We also recall that B_0 is the constant in the assumption (1.3).

Theorem 1.6. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Fix $o \in M$ and set $\rho(\cdot) = d(o, \cdot)$, where d is the Riemannian distance in M . Assume that*

$$(1.10) \quad -(b \circ \rho)^2(x) \leq \text{Sect}_x(P) \leq -(a \circ \rho)^2(x)$$

for all $x \in M$ and all 2-dimensional subspaces $P \subset T_x M$. Then the asymptotic Dirichlet problem for the equation (1.1) is uniquely solvable for any boundary data $f \in C(M(\infty))$ whenever $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$.

Observe that $B_0 = 0$ for the minimal graph equation $\mathcal{M}[u] = 0$, and therefore the condition $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$ is satisfied in Theorem 1.5. On the other hand, in the case of the p -Laplacian this condition reads as $1 < p < (n-1)\phi + 1$ and it is known to be sharp; see [32, Example 2].

Another special case, where the curvature is bounded from above by a negative constant $-k^2$, generalizes Theorem 1.4 and gives another proof for the above mentioned result of Ripoll and Telichevesky [25, Theorem 14]. Here no further restriction for the constant B_0 is needed. The validity of the assumptions (A1)-(A7) for the curvature bounds in Theorem 1.5 and Corollary 1.7 are verified in Examples 2.3 and 2.4, respectively.

Corollary 1.7. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Fix $o \in M$ and set $\rho(\cdot) = d(o, \cdot)$, where d is the Riemannian distance in M . Assume that*

$$(1.11) \quad -\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq \text{Sect}_x(P) \leq -k^2$$

for some constants $k > 0$ and $\varepsilon > 0$ and for all $x \in M \setminus B(o, R_0)$. Then the asymptotic Dirichlet problem for the equation (1.1) is uniquely solvable for any boundary data $f \in C(M(\infty))$.

We close this introduction with comments on the necessity of curvature bounds. It is worth of pointing out that the curvature bounds used in this paper are essentially the most general ones under which the asymptotic Dirichlet problem is known to be solvable, for instance, for the usual Laplace equation ([22]), for the p -Laplace equation or the \mathcal{A} -harmonic equation ([21], [33]), or for the minimal graph equation ([25] and the current paper). On the other hand, Ancona's and Borbély's

examples ([4], [8]) show that a (strictly) negative curvature upper bound alone is not sufficient for the solvability of the asymptotic Dirichlet problem for the Laplace equation. In [18], Holopainen generalized Borbély's result to the p -Laplace equations, and very recently, Holopainen and Ripoll [20] extended these nonsolvability results to equations (1.1), in particular, to the minimal graph equation.

2. PRELIMINARIES

In this section we introduce the assumptions for the curvature bounds and consider the settings in Theorem 1.5 and Corollary 1.7 as examples.

We start with the following *Comparison principle* that is crucial for the rest of the paper. Although its short proof follows the ideas in [17, Lemma 3.18] (see also [25, Lemma 3]) we feel it appropriate to give the details.

Lemma 2.1. *If $u \in W^{1,p}(\Omega)$ is a \mathcal{Q} -supersolution and $v \in W^{1,p}(\Omega)$ is a \mathcal{Q} -subsolution such that $\varphi = \min(u - v, 0) \in W_0^{1,p}(\Omega)$, then $u \geq v$ a.e. in Ω .*

Proof. Using the non-negative function $-\varphi$ as a test function we obtain

$$\begin{aligned} 0 &\geq \int_{\Omega} \langle \mathcal{A}(|\nabla v|^2) \nabla v, -\nabla \varphi \rangle dm - \int_{\Omega} \langle \mathcal{A}(|\nabla u|^2) \nabla u, -\nabla \varphi \rangle dm \\ &= \int_{\Omega \cap \{u < v\}} \langle \mathcal{A}(|\nabla v|^2) \nabla v - \mathcal{A}(|\nabla u|^2) \nabla u, \nabla v - \nabla u \rangle dm. \end{aligned}$$

On the other hand, estimating the integrand from below by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\langle \mathcal{A}(|\nabla v|^2) \nabla v - \mathcal{A}(|\nabla u|^2) \nabla u, \nabla v - \nabla u \rangle \\ &\geq \mathcal{A}(|\nabla v|^2) |\nabla v|^2 - \mathcal{A}(|\nabla v|^2) |\nabla v| |\nabla u| - \mathcal{A}(|\nabla u|^2) |\nabla u| |\nabla v| + \mathcal{A}(|\nabla u|^2) |\nabla u|^2 \\ &= (|\nabla v| \mathcal{A}(|\nabla v|^2) - |\nabla u| \mathcal{A}(|\nabla u|^2)) (|\nabla v| - |\nabla u|) \geq 0, \end{aligned}$$

where the last inequality holds since $t \mapsto t\mathcal{A}(t^2)$ is increasing. Hence the non-negative integrand must vanish a.e. in $\Omega \cap \{u < v\}$. Furthermore, since $t \mapsto t\mathcal{A}(t^2)$ is strictly increasing, we have $|\nabla u| = |\nabla v|$ a.e. in $\Omega \cap \{u < v\}$, but then

$$0 = \langle \mathcal{A}(|\nabla v|^2) \nabla v - \mathcal{A}(|\nabla u|^2) \nabla u, \nabla v - \nabla u \rangle = \mathcal{A}(|\nabla v|^2) |\nabla v - \nabla u|^2$$

a.e. in $\Omega \cap \{u < v\}$, and so $\nabla \varphi = 0$ a.e. in $\Omega \cap \{u < v\}$. Because $\varphi \in W_0^{1,p}(\Omega)$, we finally have $\varphi = 0$ a.e. in Ω and the claim follows. \square

As a consequence, we obtain the uniqueness of \mathcal{Q} -solutions with fixed (Sobolev) boundary data.

Corollary 2.2. *If $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p}(\Omega)$ are \mathcal{Q} -solutions with $u - v \in W_0^{1,p}(\Omega)$, then $u = v$ a.e. in Ω .*

We will use extensively various estimates obtained in [21] (and originated in the unpublished licentiate thesis [31]). Therefore for readers' convenience we use basically the same notation as in [21]. Thus we let M be a Cartan-Hadamard manifold, $M(\infty)$ the sphere at infinity, and $\bar{M} = M \cup M(\infty)$. Recall that the sphere at infinity is defined as the set of all equivalence classes of unit speed geodesic rays in M ; two such rays γ_1 and γ_2 are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y} : \mathbb{R} \rightarrow M$

such that $\gamma_0^{x,y} = x$ and $\gamma_t^{x,y} = y$ for some $t \in (0, \infty]$. If $v \in T_x M \setminus \{0\}$, $\alpha > 0$, and $r > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} : \angle(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, r) = C(v, \alpha) \setminus \bar{B}(x, r),$$

where $\angle(v, \dot{\gamma}_0^{x,y})$ is the angle (taken in $[0, \pi]$) between vectors v and $\dot{\gamma}_0^{x,y}$ in $T_x M$. The sets of all cones and all open balls in M form a basis for the cone topology on \bar{M} .

Throughout the paper we assume that sectional curvatures of M are bounded both from above and below by

$$(2.1) \quad -(b \circ \rho)^2(x) \leq \text{Sect}_x(P) \leq -(a \circ \rho)^2(x), \quad \rho(x) = d(x, o),$$

for all $x \in M$ and all 2-dimensional subspaces $P \subset T_x M$. Here a and b are smooth functions $[0, \infty) \rightarrow [0, \infty)$ that are constant in some neighborhood of 0 and $b \geq a$. Furthermore, we assume that b is monotonic and that there exist constants $T_1, C_1, C_2, C_3 > 0$, and $Q \in (0, 1)$ such that

$$(A1) \quad a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \geq C_1 t^{-1} & \text{if } b \text{ is increasing} \end{cases}$$

for all $t \geq T_1$ and

$$(A2) \quad a(t) \leq C_2,$$

$$(A3) \quad b(t+1) \leq C_2 b(t),$$

$$(A4) \quad b(t/2) \leq C_2 b(t),$$

$$(A5) \quad b(t) \geq C_3 (1+t)^{-Q}$$

for all $t \geq 0$. In addition, we assume that

$$(A6) \quad \lim_{t \rightarrow \infty} \frac{b'(t)}{b(t)^2} = 0$$

and that there exists a constant $C_4 > 0$ such that

$$(A7) \quad \lim_{t \rightarrow \infty} \frac{t^{1+C_4} b(t)}{f'_a(t)} = 0.$$

Here the function f_a is the solution of the Jacobi equation $f''_a = a^2 f_a$; see (2.2).

The curvature bounds are needed to control first and second order derivatives of certain "barrier" functions that will be constructed in the next section. To this end, if $k: [0, \infty) \rightarrow [0, \infty)$ is a smooth function, we denote by $f_k \in C^\infty([0, \infty))$ the solution to the initial value problem

$$(2.2) \quad \begin{cases} f_k(0) = 0, \\ f'_k(0) = 1, \\ f''_k = k^2 f_k. \end{cases}$$

It follows that the solution f_k is a non-negative smooth function.

We close this section with two examples where we verify that the curvature bounds that appear in Theorem 1.5 and Corollary 1.7 satisfy the assumption (A1)-(A7).

Example 2.3. As a first example we consider the curvature bounds in Theorem 1.5. Write $C_1 = \sqrt{\phi(\phi - 1)}$. For $t \geq R_0$ let

$$a(t) = \frac{C_1}{t}$$

and

$$b(t) = t^{\phi-2-\varepsilon/2},$$

where $0 < \varepsilon < 2\phi - 2$, and extend them to smooth functions $a: [0, \infty) \rightarrow (0, \infty)$ and $b: [0, \infty) \rightarrow (0, \infty)$ such that they are constants in some neighborhood of 0, b is monotonic and $b \geq a$. This is possible since

$$C_1 t^{-1} \leq t^{\phi-2-\varepsilon/2}$$

for $t \geq R_0$ by the curvature assumption (1.9). It is easy to verify that then

$$f_a(t) = c_1 t^{\phi_1} + c_2 t^{1-\phi_1}$$

for all $t \geq R_0$, where

$$\phi_1 = \frac{1 + \sqrt{1 + 4C_1^2}}{2} > 1,$$

$$c_1 = R_0^{-\phi_1} \frac{f_a(R_0)(\phi_1 - 1) + R_0 f'_a(R_0)}{2\phi_1 - 1} > 0,$$

and

$$c_2 = R_0^{\phi_1-1} \frac{f_a(R_0)\phi_1 - R_0 f'_a(R_0)}{2\phi_1 - 1}.$$

We then have

$$\lim_{t \rightarrow \infty} \frac{t f'_a(t)}{f_a(t)} = \phi_1$$

and, for all $C_4 \in (0, \varepsilon/2)$

$$\lim_{t \rightarrow \infty} \frac{t^{1+C_4} b(t)}{f'_a(t)} = 0.$$

It follows that a and b satisfy (A1)-(A7) with constants $T_1 = R_0$, C_1 , some $C_2 > 0$, some $C_3 > 0$, $Q = \max\{1/2, -\phi + 2 + \varepsilon/2\}$, and any $C_4 \in (0, \varepsilon/2)$.

Example 2.4. Let $k > 0$ and $\varepsilon > 0$ be constants and define $a(t) = k$ for all $t \geq 0$. Define

$$b(t) = t^{-1-\varepsilon/2} e^{kt}$$

for $t \geq R_0 = r_0 + 1$, where $r_0 > 0$ is so large that $t \mapsto t^{-1-\varepsilon/2} e^{kt}$ is increasing and greater than k for all $t \geq r_0$. Extend b to an increasing smooth function $b: [0, \infty) \rightarrow [k, \infty)$ that is constant in some neighborhood of 0. Now we can choose $C_1 > 0$ in (A1) as large as we wish. In particular, once the operator \mathcal{A} and hence the constant B_0 is chosen, we may fix C_1 so large that

$$\phi_1 = \frac{1 + \sqrt{1 + 4C_1^2}}{2}$$

satisfies $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$. Then a and b satisfy (A1)-(A7) with constants C_1 , $T_1 = C_1/k$, some $C_2 > 0$, some $C_3 > 0$, $Q = 1/2$, and any $C_4 \in (0, \varepsilon/2)$.

3. CONSTRUCTION OF A BARRIER

To solve the asymptotic Dirichlet problem for \mathcal{Q} with given continuous boundary data $f \in C(M(\infty))$, the first task is to construct a "barrier" for each boundary point $x_0 \in M(\infty)$. For that purpose let $v_0 = \dot{\gamma}_0^{o, x_0}$ be the initial (unit) vector of the geodesic ray γ^{o, x_0} from a fixed point $o \in M$ and define a function $h : M(\infty) \rightarrow \mathbb{R}$,

$$(3.1) \quad h(x) = \min(1, L\angle(v_0, \dot{\gamma}_0^{o, x})),$$

where $L \in (8/\pi, \infty)$ is a constant.

Next step is to extend h to a function $h \in C^\infty(M) \cap C(\bar{M})$ with controlled first and second order derivatives. This is done in [21] by defining first a crude extension $\tilde{h} : \bar{M} \rightarrow \mathbb{R}$,

$$(3.2) \quad \tilde{h}(x) = \min\left(1, \max(2 - 2\rho(x), L\angle(v_0, \dot{\gamma}_0^{o, x}))\right).$$

Then $\tilde{h} \in C(\bar{M})$ and $\tilde{h}|_{M(\infty)} = h$. As the final step in the construction of a barrier we smooth out \tilde{h} to get an extension $h \in C^\infty(M) \cap C(\bar{M})$. To this end, we fix $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\text{spt } \chi \subset [-2, 2]$, and $\chi|_{[-1, 1]} \equiv 1$. Then for any function $\varphi \in C(M)$ we define functions $F_\varphi : M \times M \rightarrow \mathbb{R}$, $\mathcal{R}(\varphi) : M \rightarrow M$, and $\mathcal{P}(\varphi) : M \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_\varphi(x, y) &= \chi(b(\rho(y))d(x, y))\varphi(y), \\ \mathcal{R}(\varphi)(x) &= \int_M F_\varphi(x, y)dm(y), \quad \text{and} \\ \mathcal{P}(\varphi) &= \frac{\mathcal{R}(\varphi)}{\mathcal{R}(1)}, \end{aligned}$$

where

$$\mathcal{R}(1) = \int_M \chi(b(\rho(y))d(x, y))dm(y) > 0.$$

Thus $\mathcal{P}(\varphi)$ is an integral average of φ with respect to χ similar to that in [6, p. 436] except that here the function b is taken into account explicitly. If $\varphi \in C(\bar{M})$, we extend $\mathcal{P}(\varphi) : M \rightarrow \mathbb{R}$ to a function $\bar{M} \rightarrow \mathbb{R}$ by setting $\mathcal{P}(\varphi)(x) = \varphi(x)$ whenever $x \in M(\infty)$. Then the extended function $\mathcal{P}(\varphi)$ is C^∞ -smooth in M and continuous in \bar{M} ; see [21, Lemma 3.13]. In particular, applying \mathcal{P} to the function \tilde{h} yields an appropriate smooth extension

$$(3.3) \quad h := \mathcal{P}(\tilde{h})$$

of the original function $h \in C(M(\infty))$ that was defined in (3.1).

We obtain control on first and second order derivatives of the extended function h from the curvature assumption (2.1) by the Rauch and Hessian comparison theorems. Here the solutions f_a and f_b to the initial value problem (2.2), where a and b are curvature bounds in (2.1) satisfying (A1)-(A7), play an important role. Another crucial point is that the mollifying procedure above depends on the curvature lower bound function b . For the next lemma and later purposes we denote

$$\Omega = C(v_0, 1/L) \cap M \quad \text{and} \quad k\Omega = C(v_0, k/L) \cap M$$

for $k > 0$. We collect together all these constants and functions and denote

$$C = (a, b, T_1, C_1, C_2, C_3, C_4, Q, n, L).$$

Furthermore, we denote by $\|\text{Hess}_x u\|$ the norm of the Hessian of a smooth function u at x , that is

$$\|\text{Hess}_x u\| = \sup_{\substack{X \in T_x M \\ |X| \leq 1}} |\text{Hess } u(X, X)|.$$

The following lemma provides an important estimate.

Lemma 3.1. [21, Lemma 3.16] *There exist constants $R_1 = R_1(C)$ and $c_5 = c_5(C)$ such that the extended function $h \in C^\infty(M) \cap C(\bar{M})$ in (3.3) satisfies*

$$(3.4) \quad \begin{aligned} |\nabla h(x)| &\leq c_5 \frac{1}{(f_a \circ \rho)(x)}, \\ \|\text{Hess}_x h\| &\leq c_5 \frac{(b \circ \rho)(x)}{(f_a \circ \rho)(x)}, \end{aligned}$$

for all $x \in 3\Omega \setminus B(o, R_1)$. In addition,

$$h(x) = 1$$

for every $x \in M \setminus (2\Omega \cup B(o, R_1))$.

Let then $A > 0$ be a fixed constant. We aim to show that

$$(3.5) \quad \varphi = A(R_4^\delta \rho^{-\delta} + h)$$

is a \mathcal{Q} -supersolution in the set $3\Omega \setminus \bar{B}(o, R_4)$, where $\delta > 0$ and $R_4 > 0$ are constants that will be specified later and h is the extended function defined in (3.3). First of all φ is C^∞ -smooth in $M \setminus \{o\}$ and there

$$\nabla \varphi = A(-R_4^\delta \delta \rho^{-\delta-1} \nabla \rho + \nabla h)$$

and

$$\Delta \varphi = A(R_4^\delta \delta(\delta+1) \rho^{-\delta-2} - R_4^\delta \delta \rho^{-\delta-1} \Delta \rho + \Delta h).$$

We shall make use of the following estimates obtained in [21]; see also [19]:

Lemma 3.2. [21, Lemma 3.17] *There exist constants $R_2 = R_2(C)$ and $c_6 = c_6(C)$ with the following property. If $\delta \in (0, 1)$, then*

$$\begin{aligned} |\nabla h| &\leq c_6 / (f_a \circ \rho), \\ \|\text{Hess } h\| &\leq c_6 \rho^{-C_4-1} (f'_a \circ \rho) / (f_a \circ \rho), \\ |\nabla \langle \nabla h, \nabla h \rangle| &\leq c_6 \rho^{-C_4-2} (f'_a \circ \rho) / (f_a \circ \rho), \\ |\nabla \langle \nabla h, \nabla(\rho^{-\delta}) \rangle| &\leq c_6 \rho^{-C_4-2} (f'_a \circ \rho) / (f_a \circ \rho), \\ \nabla \langle \nabla(\rho^{-\delta}), \nabla(\rho^{-\delta}) \rangle &= -2\delta^2(\delta+1) \rho^{-2\delta-3} \nabla \rho \end{aligned}$$

in the set $3\Omega \setminus B(o, R_2)$.

As in [21] we denote

$$\phi_1 = \frac{1 + \sqrt{1 + 4C_1^2}}{2} > 1, \quad \text{and} \quad \delta_1 = \min \left\{ C_4, \frac{-1 + (n-1)\phi_1}{1 + (n-1)\phi_1} \right\} \in (0, 1),$$

where C_1 and C_4 are constants from (A1) and (A7), respectively. Then by [21, Lemma 3.18] there exists $R_3 = R_3(C, \delta) \geq R_2$ such that

$$(3.6) \quad -\Delta(\rho^{-\delta}) > 0 \quad \text{and} \quad \frac{|\Delta h|}{-\Delta(\rho^{-\delta})} \leq \delta$$

in $3\Omega \setminus B(o, R_3)$.

Suppose then that

$$B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$$

and let $0 < \delta < \min(\delta_1, \phi_1 - 1, C_4/2)$ be so small that

$$(3.7) \quad \delta + \frac{2\lambda(\max(0, B_0) + \bar{B}_0\delta)}{(1-\lambda)(1-\delta)^3} < 1,$$

where B_0 is the constant in (1.3), $\bar{B}_0 = \max(\frac{1}{2}, B_0)$, and

$$\lambda = \frac{1+\delta}{(1-\delta)(n-1)\phi_1} \in (0, 1).$$

Such δ exists because $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$. Then there exists $R_4 = R_4(C, B_0) \geq \min(R_3, 1)$ such that, in addition to estimates in (3.6), we have

$$(3.8) \quad \frac{-\Delta(\rho^{-\delta})}{\delta\rho^{-\delta-1}\Delta\rho} \geq 1 - \lambda, \quad \frac{|\nabla h|}{|\nabla(\rho^{-\delta})|} \leq \delta, \quad \frac{\rho(f'_a \circ \rho)}{f_a \circ \rho} \geq (1-\delta)\phi_1,$$

and

$$(3.9) \quad \frac{3\bar{B}_0 c_6 \rho^{-C_4+2\delta}}{R_4^\delta (1-\delta/R_4^\delta)^2 \delta^2 (1-\lambda)(n-1)} \leq \delta$$

in $3\Omega \setminus B(o, R_4)$; see [21, Lemma 3.18, (3.30), (3.32)] for the estimates in (3.8). The estimate (3.9) is possible because $-C_4 + 2\delta < 0$.

We are now in a position to prove that $\varphi = A(R_4^\delta \rho^{-\delta} + h)$, for any given constant $A > 0$, is a \mathcal{Q} -supersolution in an open truncated cone $3\Omega \setminus \bar{B}(o, R_4)$ whenever $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$. As a smooth function φ is a \mathcal{Q} -supersolution if $\operatorname{div} \mathcal{A}(|\nabla \varphi|^2) \nabla \varphi \leq 0$. On the other hand, it follows from (3.6) and (3.8) that

$$|\nabla \varphi| > 0 \quad \text{and} \quad \Delta \varphi < 0$$

in $3\Omega \setminus B(o, R_4)$. Hence we can write

$$\begin{aligned} \operatorname{div} \mathcal{A}(|\nabla \varphi|^2) \nabla \varphi &= \mathcal{A}(|\nabla \varphi|^2) \Delta \varphi + \mathcal{A}'(|\nabla \varphi|^2) \langle \nabla \langle \nabla \varphi, \nabla \varphi \rangle, \nabla \varphi \rangle \\ &= \mathcal{A}(|\nabla \varphi|^2) \{ \Delta \varphi + \mathcal{B}(|\nabla \varphi|^2) \langle \nabla \langle \nabla \varphi, \nabla \varphi \rangle, \nabla \varphi \rangle \} \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$. Therefore φ is a \mathcal{Q} -supersolution if

$$(3.10) \quad \frac{\mathcal{B}(|\nabla \varphi|^2) |\nabla \varphi|^2 \langle \nabla \langle \nabla \varphi, \nabla \varphi \rangle, \nabla \varphi \rangle}{-|\nabla \varphi|^2 \Delta \varphi} < 1$$

in $3\Omega \setminus \bar{B}(o, R_4)$.

Lemma 3.3. *Let $A > 0$ be a fixed constant and let h be the function defined in (3.3). Then there exist constants $\delta = \delta(C, B_0) \in (0, \delta_1)$ and $R_4 = R_4(C, B_0)$ such that the function $\varphi = A(R_4^\delta \rho^{-\delta} + h)$ is a \mathcal{Q} -supersolution in the set $3\Omega \setminus \bar{B}(o, R_4)$ whenever $B_0 < \frac{1}{2}((n-1)\phi_1 - 1)$.*

Proof. Since all estimates in this proof are made in the set $3\Omega \setminus B(o, R_4)$, we do not indicate this all the time. Writing $u = R_4^\delta \rho^{-\delta} + h$ we have

$$\begin{aligned}
\frac{\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2\langle\nabla\langle\nabla\varphi, \nabla\varphi\rangle, \nabla\varphi\rangle}{-|\nabla\varphi|^2\Delta\varphi} &= \frac{\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2\langle\nabla\langle\nabla u, \nabla u\rangle, \nabla u\rangle}{-|\nabla u|^2\Delta u} \\
&= \frac{\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2}{-|\nabla u|^2\Delta u} \left(R_4^{3\delta}\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla(\rho^{-\delta})\rangle \right. \\
&\quad \left. + \langle\nabla\langle\nabla h, \nabla h\rangle + 2R_4^\delta\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle, \nabla u\rangle + R_4^{2\delta}\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla h\rangle \right) \\
&\leq \frac{\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2 R_4^{3\delta}\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla(\rho^{-\delta})\rangle}{-|\nabla u|^2\Delta u} \\
&\quad + \frac{\bar{B}_0(|\nabla\langle\nabla h, \nabla h\rangle| + 2R_4^\delta|\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle|)}{-|\nabla u|\Delta u} + \frac{\bar{B}_0 R_4^{2\delta}|\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle||\nabla h|}{-|\nabla u|^2\Delta u}.
\end{aligned}$$

We estimate the three terms above separately. By the standard Laplace comparison theorem (see e.g. [21, Prop. 2.5(b)]) and (3.8) we have

$$(3.11) \quad \Delta\rho \geq (n-1)\frac{f'_a \circ \rho}{f_a \circ \rho} \geq \frac{(n-1)(1-\delta)\phi_1}{\rho}.$$

As in [21], we denote

$$T = \frac{|\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle|}{-|\nabla u|\Delta u} = \frac{2\delta^2(\delta+1)\rho^{-2\delta-3}}{-|\nabla u|\Delta u}.$$

Using (3.6), (3.8), and (3.11), we first obtain

$$\begin{aligned}
-|\nabla u|\Delta u &\geq -(R_4^\delta - \delta)^2|\nabla(\rho^{-\delta})|\Delta(\rho^{-\delta}) \\
&\geq (R_4^\delta - \delta)^2\delta^2(1-\lambda)\rho^{-2\delta-2}\Delta\rho \\
(3.12) \quad &\geq (R_4^\delta - \delta)^2\delta^2(1-\lambda)\rho^{-2\delta-2}(n-1)\frac{f'_a \circ \rho}{f_a \circ \rho} \\
&\geq (R_4^\delta - \delta)^2\delta^2(1-\lambda)\rho^{-2\delta-3}(n-1)(1-\delta)\phi_1,
\end{aligned}$$

and therefore

$$(3.13) \quad T = \frac{2\delta^2(\delta+1)\rho^{-2\delta-3}}{-|\nabla u|\Delta u} \leq \frac{2\lambda}{(R_4^\delta - \delta)^2(1-\lambda)}.$$

Since

$$\frac{\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla(\rho^{-\delta})\rangle}{-|\nabla u|^2\Delta u} = \frac{2\delta^3(\delta+1)\rho^{-3\delta-4}}{-|\nabla u|^2\Delta u} > 0,$$

we can estimate the first term as

$$\begin{aligned}
&\frac{R_4^{3\delta}\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla(\rho^{-\delta})\rangle}{-|\nabla u|^2\Delta u} \\
(3.14) \quad &\leq \frac{\max(0, B_0)R_4^{3\delta}T|\nabla(\rho^{-\delta})|}{|\nabla u|} \\
&\leq \frac{2\max(0, B_0)\lambda}{(1-\delta/R_4^\delta)^3(1-\lambda)} \\
&\leq \frac{2\max(0, B_0)\lambda}{(1-\delta)^3(1-\lambda)}.
\end{aligned}$$

The second term can be estimated by Lemma 3.2, (3.9), and (3.12) as

$$\begin{aligned}
 (3.15) \quad & \frac{\bar{B}_0(|\nabla\langle\nabla h, \nabla h\rangle| + 2R_4^\delta|\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle|)}{-|\nabla u|\Delta u} \\
 & \leq \frac{\bar{B}_0(|\nabla\langle\nabla h, \nabla h\rangle| + 2R_4^\delta|\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle|)(f_a \circ \rho)}{(R_4^\delta - \delta)^2\delta^2\rho^{-2\delta-2}(1-\lambda)(n-1)(f'_a \circ \rho)} \\
 & \leq \frac{\bar{B}_0(1 + 2R_4^\delta)c_6\rho^{-C_4+2\delta}}{(R_4^\delta - \delta)^2\delta^2(1-\lambda)(n-1)} \\
 & \leq \frac{3\bar{B}_0c_6\rho^{-C_4+2\delta}}{R_4^\delta(1-\delta/R_4^\delta)^2\delta^2(1-\lambda)(n-1)} \leq \delta.
 \end{aligned}$$

The third term can be estimated by using (3.8) and (3.13) as

$$\begin{aligned}
 (3.16) \quad & \frac{\bar{B}_0R_4^{2\delta}|\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle||\nabla h|}{-|\nabla u|^2\Delta u} = \frac{\bar{B}_0R_4^{2\delta}T|\nabla h|}{|\nabla u|} \\
 & \leq \frac{2\bar{B}_0R_4^{2\delta}\delta\lambda}{(R_4^\delta - \delta)^3(1-\lambda)} \\
 & \leq \frac{2\bar{B}_0\delta\lambda}{(1-\delta)^3(1-\lambda)}.
 \end{aligned}$$

Putting the estimates (3.14)-(3.16) and (3.7) together we finally obtain

$$\begin{aligned}
 & \frac{\mathcal{B}(|\nabla\varphi|^2)|\nabla\varphi|^2\langle\nabla\langle\nabla\varphi, \nabla\varphi\rangle, \nabla\varphi\rangle}{-|\nabla\varphi|^2\Delta\varphi} \\
 & \leq \frac{2\max(0, B_0)\lambda}{(1-\delta)^3(1-\lambda)} + \delta + \frac{2\bar{B}_0\delta\lambda}{(1-\delta)^3(1-\lambda)} \\
 & \leq \delta + \frac{2\lambda(\max(0, B_0) + \bar{B}_0\delta)}{(1-\lambda)(1-\delta)^3} < 1
 \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$. Hence $\varphi = A(R_r^\delta\rho^{-\delta} + h)$ is a continuous \mathcal{Q} -supersolution in $3\Omega \setminus \bar{B}(o, R_4)$. \square

4. PROOF OF THEOREM 1.6

Let $\tilde{f} \in C(\bar{M})$ be an extension of the given boundary data $f \in C(M(\infty))$. Choose an exhaustion of M by an increasing sequence of \mathcal{Q} -regular domains Ω_k provided by the assumption (A). Hence there exist \mathcal{Q} -solutions $u_k \in C(\bar{\Omega}_k) \cap W_{\text{loc}}^{1,p}(\Omega_k)$ such that

$$\begin{cases} \mathcal{Q}[u_k] = 0 & \text{in } \Omega_k, \\ u_k|_{\partial\Omega_k} = \tilde{f}. \end{cases}$$

Then

$$-\max|\tilde{f}| \leq u_k \leq \max|\tilde{f}|$$

in Ω_k by the Comparison principle (Lemma 2.1). Condition (B) together with a diagonal argument implies that there exists a subsequence, still denoted by u_k , that converges locally uniformly in M to a \mathcal{Q} -solution $u \in C(M)$. Therefore the proof of Theorem 1.6 reduces to prove that u extends continuously to $M(\infty)$, satisfies $u|M(\infty) = f$, and is the unique \mathcal{Q} -solution with boundary values f . To this end,

let $x_0 \in M(\infty)$ and $\varepsilon > 0$. Since f is continuous, there exists $L \in (8/\pi, \infty)$ such that

$$|f(y) - f(x_0)| < \varepsilon/2$$

for all $y \in C(v_0, 4/L) \cap M(\infty)$, where $v_0 = \dot{\gamma}_0^{o, x_0}$ is the initial vector of the geodesic ray representing x_0 . We claim that

$$(4.1) \quad w(x) := -\varphi(x) + f(x_0) - \varepsilon \leq u(x) \leq v(x) := \varphi(x) + f(x_0) + \varepsilon$$

in $U = 3\Omega \setminus \bar{B}(o, R_4)$, where $\varphi = A(R_4^\delta \rho^{-\delta} + h)$ is the \mathcal{Q} -supersolution in U as in Lemma 3.3, with $A = 2 \max_{\bar{M}} |\tilde{f}|$. Note that $-\varphi$ is a \mathcal{Q} -subsolution in U . Recall the notation $\Omega = C(v_0, 1/L) \cap M$ and $k\Omega = C(v_0, k/L) \cap M$, $k > 0$, from Section 2. Since \tilde{f} is continuous in \bar{M} , there exists k_0 such that

$$(4.2) \quad |\tilde{f}(x) - f(x_0)| < \varepsilon/2$$

for all $x \in \partial\Omega_k \cap U$ and all $k \geq k_0$ and that $\partial\Omega_{k_0} \cap U \neq \emptyset$. Let $V_k = \Omega_k \cap U$ for $k \geq k_0$. We have

$$\partial V_k = (\partial\Omega_k \cap \bar{U}) \cup (\partial U \cap \bar{\Omega}_k).$$

Next we will show by using the Comparison principle that

$$(4.3) \quad w \leq u_k \leq v$$

in V_k . By (4.2), we have

$$w(x) \leq f(x_0) - \varepsilon/2 \leq \tilde{f}(x) = u_k(x) \leq f(x_0) + \varepsilon/2 \leq v(x)$$

for all $x \in \partial\Omega_k \cap \bar{U}$ and $k \geq k_0$. On the other hand,

$$h|M \setminus (2\Omega \cup B(o, R_1)) = 1$$

by Lemma 3.1 and $R_4^\delta \rho^{-\delta} = 1$ on $\partial B(o, R_4)$, and therefore $\varphi \geq A = 2 \max_{\bar{M}} |\tilde{f}|$ on $\partial U \cap \bar{\Omega}_k$. It follows that

$$v = \varphi + f(x_0) + \varepsilon \geq 2 \max_{\bar{M}} |\tilde{f}| + f(x_0) + \varepsilon \geq \max_{\bar{M}} |\tilde{f}| + \varepsilon \geq u_k$$

on $\partial U \cap \bar{\Omega}_k$. Similarly, $u_k \geq w$ on $\partial U \cap \bar{\Omega}_k$. Thus $w \leq u_k \leq v$ on ∂V_k and (4.3) follows. Since this holds for all $k \geq k_0$, we obtain (4.1). Finally,

$$\limsup_{x \rightarrow x_0} |u(x) - f(x_0)| \leq \varepsilon$$

since $\lim_{x \rightarrow x_0} \varphi(x) = 0$. Thus u extends continuously to $C(\bar{M})$ and $u|M(\infty) = f$ since $x_0 \in M(\infty)$ and $\varepsilon > 0$ were arbitrary. We are left with the uniqueness of u . Therefore, let $\tilde{u} \in C(\bar{M})$ be another \mathcal{Q} -solution in M , with $\tilde{u} = u = f$ in $M(\infty)$. Suppose on the contrary that $\tilde{u} \neq u$. Thus we may assume without loss of generality that $\tilde{u}(x) > u(x) + \varepsilon$ for some $x \in M$ and $\varepsilon > 0$. Let D be the x -component of the set $\{y \in M : \tilde{u}(y) > u(y) + \varepsilon\}$. Then D is open with compact closure since both \tilde{u} and u are continuous in \bar{M} and coincide on $M(\infty)$. Furthermore, $\tilde{u} = u + \varepsilon$ on ∂D , and therefore $\tilde{u} = u + \varepsilon$ in D by Corollary 2.2 which leads to a contradiction with $\tilde{u}(x) > u(x) + \varepsilon$. This concludes the proof of Theorem 1.6.

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